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Nonlinear Schrödinger solitons in the presence of an external potential

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Abstract. We study the influence of external potentials on the solitary wave solutions of certain nonlinear partial differential equations common in physics. Our approach allows for both the translation of pulse-like envelopes through space and the 'breathing' of these pulses in the centre-of-mass frame. We find that for certain simple potentials it is possible for the familiar soliton solutions of these equations to be preserved and to execute essentially classical motion. For more general potentials, however, we find that the familiar pulse shapes cannot be preserved; shape deformations more complex than our simple breathing motion are required. When shape deformations are not too severe, our approach allows an approximate solution for the case of adiabatic following.

1. Introduction

Recent years have seen a considerable growth in the study of nonlinear partial differential equations in physics and related fields. The nonlinear Schrödinger equation (NLSE) and its variants appear in problems drawn from disciplines as diverse as solid-state [1–3], particle [4, 5] and plasma physics [6]. The 'soliton' solutions of these equations have well defined pulsate shapes and remarkable stability properties [7]. A problem of current interest is that of determining what effect external potentials may have on the properties of these pulse-like states. Chen and Liu [8], for example, studied the propagation of electromagnetic waves in a linearly inhomogeneous plasma, described by a cubic nonlinear Schrödinger equation (CNLSE) and showed that solitons are accelerated as Newtonian particles and maintain their shape and identity even upon emerging from collisions with other solitons. Bialynicki-Birula and Mycielski [9] later showed that gaussons, solutions of the logarithmic nonlinear Schrödinger equation (LNLSE), have a classical motion for their centres of mass in the presence of a uniform electromagnetic field. A few years later, Fernandez *et al* [10] showed that the velocity of sine–Gordon kinks in a constant external field increases for small times, with the third power of time, and approaches a constant value for large times. Since this is in contrast to the classical law of a linearly increasing velocity, they refer to the sine–Gordon kinks as non-Newtonian particles. Later, Hasse [11] showed that in a constant external field, the solitons of the cubic, logarithmic, derivative, and other NLSE move with classical velocity and have the same shape as in the absence of the external field; these conclusions were limited, however, to the constant-field case. Recently, Nassar [12], using stochastic mechanics to solve the LNLSE in a time-dependent forced-harmonic-oscillator potential $[V(x,t) = \frac{1}{2}\omega^2(t)x^2 - xF(t)]$, showed that the Gaussian-shaped soliton has a classical trajectory in the sense noted above and is non-spreading only for $\omega(t) = \omega_0$ (constant). Recently, de Moura [13] treated Nassar's condition from another point of view

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and concluded that the shape of NLSE solitons is not affected by the presence of an external field.

In this paper, we study the problem of nonlinear Schrödinger solitons in external potentials from a more general point of view. We use the fact that the centre of mass of the soliton is governed by an Ehrenfest equation and we use soliton boundary conditions to obtain consistency conditions which allow us to limit the types of solutions possible and characterize their behaviour in a few simple external potentials. To illustrate the possibilities, we choose the nonlinear potential to be the usual cubic nonlinearity, leading to the CNLSE, and the logarithmic nonlinearity, leading to the LNLSE.

2. Preliminaries

There are many NLSEs in physics, many of which take the general form

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\tfrac{1}{2}\frac{\partial^2\psi}{\partial x^2} + [V(x,t) + W(|\psi|^2)]\psi \tag{1}$$

where we let $\hbar = m = 1$. The functions $V(x,t)$ and $W(|\psi|^2)$ represent the external potential and the strength of the nonlinearity, respectively. The complex 'wavefunction' $\psi(x,t)$ may represent a probability amplitude, the amplitude of an electric field or the velocity field of a fluid, for example. It is useful to separate the amplitude and phase components of this complex function and, thus, we define

$$\psi(x,t) \equiv \phi(x,t)\exp[\mathrm{i}S(x,t)]. \tag{2}$$

Consider the NLSE (1) and split it into real and imaginary parts

$$\dot{S} + W(\phi) + V(x,t) + \tfrac{1}{2}(S')^2 = \frac{\phi''}{2\phi} \tag{3}$$

and

$$\dot{\phi} + \phi' S' + \tfrac{1}{2}\phi S'' = 0 \tag{4}$$

where $\dot{\phi} = \partial\phi/\partial t$, $\phi' = \partial\phi/\partial x$, etc. We take the envelope function to have the form

$$\phi(x,t) = [\sigma(t)]^{-1/2}\tilde{\phi}\{\sigma^{-1}(t)[x - x_c(t)]\} \tag{5}$$

where $x_c(t) \equiv \langle x\rangle$ is the instantaneous centre of mass of the wavepacket and $\sigma(t)$ is the instantaneous packet width, where the average is the usual quantum expectation value

$$\langle\hat{O}\rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\hat{O}\Psi(x,t)\,\mathrm{d}x. \tag{6}$$

With this representation of the envelope function, using a time- and space-independent shape function $\tilde{\phi}$, all space dependence is explicit and all time dependences are born by the parameters $x_c(t)$ and $\sigma(t)$. We write $\phi(x,t)$ in this form so that it can be normalized to a constant value, as for the usual normalization for conserved particles. It is important to realize that since the shape function $\tilde{\phi}$ is 'rigid', with all changes in shape being given by the time dependence of the width function $\sigma(t)$, the shape, once determined, is fixed for all time and the only deformations which can occur are breathing-type motions determined by $\sigma(t)$.

The continuity equation (4) can be put in the form

$$\frac{\partial^2 S}{\partial x^2}+\left[\frac{\partial}{\partial x}\ln\phi^2\right]\frac{\partial S}{\partial x}=-\frac{\partial}{\partial t}\ln\phi^2. \tag{7}$$

This equation is simply integrated with respect to the space variable, yielding

$$\frac{\partial S(x,t)}{\partial x}=\dot{x}_c+\frac{\dot{\sigma}}{\sigma}(x-x_c)+\frac{A(t)}{\phi^2} \tag{8}$$

where $A(t)$ is a (partial) constant of integration which must be set to zero to maintain a finite total current. With this choice of $A(t)$, we may integrate once again, yielding

$$S(x,t)=\dot{x}_c x+\frac{\dot{\sigma}}{\sigma}(\tfrac{1}{2}x^2-x_c x)+B(t) \tag{9}$$

where $B(t)$ is a 'constant' of integration which can be related to the energy. Having a formal solution for $S(x,t)$, this can be inserted into (3) to yield

$$\ddot{x}_c=-\frac{\partial V(x,t)}{\partial x}+\frac{\partial}{\partial x}\left[\frac{1}{2\phi}\frac{\partial^2\phi}{\partial x^2}-W(\phi)\right]-\frac{\ddot{\sigma}}{\sigma}(x-x_c) \tag{10}$$

which for $\sigma=$ constant is the same as equation (5) of [13]. This equation relates the motion of the wavepacket centre of mass to its width, as well as to the external and 'inertial' forces acting upon it. We now take the average of (10) to obtain

$$\ddot{x}_c=-\left\langle\frac{\partial V(x,t)}{\partial x}\right\rangle+\left\langle\frac{\partial}{\partial x}\left[\frac{1}{2\phi}\frac{\partial^2\phi}{\partial x^2}-W(\phi)\right]\right\rangle. \tag{11}$$

Using the definition of the expectation value, and the identity $\int\phi_{xx}\phi_x\,\mathrm{d}x=\frac{1}{2}\phi_x^2$, we have, for the second average on the right-hand side of (11),

$$\left\langle\frac{\partial}{\partial x}\left[\frac{1}{2\phi}\frac{\partial^2\phi}{\partial x^2}-W(\phi)\right]\right\rangle=\phi^2\left[\frac{\phi_{xx}}{2\phi}-W(\phi)\right]\Big|_{-\infty}^{+\infty}-\tfrac{1}{2}\phi_x^2\Big|_{-\infty}^{+\infty}+2\int_{\phi(-\infty)}^{\phi(+\infty)}W(\phi)\phi\,\mathrm{d}\phi \tag{12}$$

which vanishes identically for soliton boundary conditions ($\phi(\pm\infty)=$ constant, $\phi_x(\pm\infty)=0$). This is easily verified for $W=-\alpha^2\phi^2$ (CNLSE) and $W=-\lambda^2\ln\phi$ (LNLSE) as well as for other common NLSEs [14]. Thus, provided that only a pulse-like solution exists, its centre of mass will follow an essentially classical trajectory; i.e. it will follow the Ehrenfest equation [15]

$$\ddot{x}_c=-\left\langle\frac{\partial V(x,t)}{\partial x}\right\rangle. \tag{13}$$

The general form for the energy of the system described by (1) can be given in the form

$$E=-\langle\dot{S}\rangle=-\ddot{x}_c x_c-\left[\frac{\ddot{\sigma}}{2\sigma}-\frac{\dot{\sigma}^2}{2\sigma^2}\right](\gamma^2-x_c^2)+\frac{\dot{\sigma}}{\sigma}\dot{x}_c x_c-\dot{B}(t) \tag{14}$$

where

$$\gamma^2 \equiv \int_{-\infty}^{+\infty} \xi^2 \tilde{\phi}^2(\xi)\, d\xi = \text{constant}. \tag{15}$$

We note that when both the velocity and width of the pulse are constant, this relation reduces to

$$E = -\dot{B}(t) \tag{16}$$

which identifies $B(t) = \hbar\omega t$ in such cases. A useful expression for the energy is also obtained by substituting (14) and (8) into the average of (3), yielding

$$E = \tfrac{1}{2}\dot{x}_c^2 + \langle V(x,t)\rangle + \frac{\gamma^2}{2}\dot{\sigma}^2 - \left\langle \frac{\phi''}{2\phi} \right\rangle + \langle W(\phi)\rangle. \tag{17}$$

In the following section, we will use only time-independent external potentials ($V(x,t) = V(x)$) and use (17) to impose conservation of energy.

3. Application to the solitons of the LNLSE and the CNLSE

In this section, we study the dynamics of solitons in some simple external potentials using the main results of section 2 from equations (10), (13) and (17).

3.1. *Constant potential ($V = 0$)*

It is trivially found from the Ehrenfest equation that $\ddot{x}_c = 0$ for any type of wavefunction (the centre of mass is unaccelerated). Inserting this into (10), the envelope equation takes the form

$$\phi_{xx} - 2W(\phi)\phi = \frac{\ddot{\sigma}}{\sigma}[(x - x_c)^2 + C(t)]\phi \tag{18}$$

where $C(t)$ is a 'constant' of integration.

For the LNLSE ($W = -\lambda^2 \ln\phi$), (18) yields the unique Gaussian solution

$$\phi(x,t) = \frac{1}{\pi^{1/4}}\frac{1}{\sigma^{1/2}} \exp\left[-(x - x_c)^2/2\sigma^2\right] \tag{19}$$

subject to the subsidiary condition

$$\dot{\sigma}\left[\ddot{\sigma} - \sigma^{-3} + \lambda^2\sigma^{-1}\right] = 0 \tag{20}$$

which may be viewed either as a consequence of boundary conditions or of conservation of energy. Any constant-width solution satisfies (20); however, a special class of internal oscillations is also possible. Since the centre-of-mass motion and internal motion are decoupled, these internal modes may be excited independently and the energy thus stored transported through space.

For the CNLSE ($W(\phi) = -\alpha^2\phi^2$), a general analytic solution comparable to the Gaussian solution found above seems unlikely. Rather than considering boundary conditions, therefore, we first consider conservation of energy. Since the 'external' kinetic- and potential-energy terms in (17) are each constant, the 'internal' energy terms are forced to sum to a constant, a fact which can be used to derive an equation governing any possible internal motions of the pulse. Setting the time derivative of the energy to zero, we find

$$\dot{\sigma}\left[\ddot{\sigma}+\frac{\alpha^2\delta^2}{\gamma^2}\frac{1}{\sigma^2}-\frac{\mu^2}{\gamma^2\sigma^3}\right]=0 \tag{21}$$

where γ^2 is defined as in (15), and

$$\delta^2=\int_{-\infty}^{+\infty}\tilde{\phi}^4(\xi)\,\mathrm{d}\xi \tag{22}$$

$$\mu^2=\int_{-\infty}^{+\infty}\tilde{\phi}'^2(\xi)\,\mathrm{d}\xi. \tag{23}$$

Considering the boundary conditions which must be applied, however, one can see that none of the oscillations which *appear* to be allowed by (21) correspond to physically meaningful solutions; thus, only constant-width pulses are allowed. For constant widths, (18) reduces to the usual CNLSE which yields the well known *sech* soliton solution.

3.2. *Constant field* ($V=-gx$)

A simple calculation using only the form of the trial state ϕ, as given in (5), shows that the 'external' kinetic- and potential-energy terms are each time-dependent, but sum to a constant. As a result, we are left with precisely the same 'internal' equations of motion as considered above, and all conclusions regarding the types of physically meaningful solutions continue to hold for the LNLSE and CNLSE, respectively. For the LNLSE, this means that the decoupling of the translational motion of the pulse and its internal oscillations persists despite the fact that the centre of mass is uniformly accelerated. For the CNLSE, this result is consistent with the findings of Chen and Liu [8] in their study of the propagation of electromagnetic disturbances in a linearly inhomogeneous plasma.

3.3. *Harmonic potential* ($V=\frac{1}{2}\omega^2x^2$)

This case presents us with perhaps the simplest example of non-uniformly accelerated motion. From the Ehrenfest equation (13), we find that the centre of mass undergoes simple harmonic motion at the natural frequency ω. The 'external' kinetic and potential energies clearly oscillate at the natural frequency as well; however, unlike the previous cases, the sum of these terms is not constant. In order for energy to be conserved, it is necessary that this fluctuation in the 'external' energy be matched by a compensating fluctuation in the internal energy. In general, therefore, the external potential will have an effect on the internal 'breathing' motion of the pulse.

For the LNLSE, insertion of the Ehrenfest equation into (10) leads to the equation of motion for internal oscillations:

$$\dot{\sigma}\left[\ddot{\sigma}+\omega^2\sigma+\frac{\lambda^2}{2\gamma^2}\frac{1}{\sigma}-\frac{\mu^2}{\gamma^2}\frac{1}{\sigma^3}\right]=0. \tag{24}$$

This is in agreement with the result of Nassar [12], obtained using stochastic mechanics. The appearance of ω in this equation is evidence of the coupling of the internal and external motion noted above. The envelope equation in the presence of this harmonic potential becomes

$$\phi_{xx} + 2\lambda^2 \ln \phi = \left[\frac{\ddot{\sigma}}{\sigma} + \omega^2\right][(x - x_c)^2 + b(t)]\phi \tag{25}$$

which has the *same* Gaussian solution as given in (19). The effect of the external potential on the envelope is indirect, appearing only through the modification of the effective potential controlling the internal motion. This produces no qualitative change, however, since the internal motion in previous examples was already oscillatory and only the detail of the internal oscillation is affected.

For the CNLSE, we may again demand energy conservation and arrive at the equation for internal oscillations

$$\dot{\sigma}\left[\ddot{\sigma} + \omega^2\sigma + \frac{\alpha^2\delta^2}{\gamma^2}\frac{1}{\sigma^2} - \frac{\mu^2}{\gamma^2\sigma^3}\right] = 0 \tag{26}$$

which is complemented by the envelope equation

$$\phi_{xx} + 2\alpha^2\phi^3 = \left[\frac{\ddot{\sigma}}{\sigma} + \omega^2\right][(x - x_c)^2 + b(t)]\phi. \tag{27}$$

By considering soliton boundary conditions, however, one can show that these equations are incompatible for *any* internal oscillation allowed by (26). Moreover, in this case, even a constant-width soliton solution cannot exist. The necessary conclusion is that the cubic nonlinearity is insufficient to preserve the shape of a pulse in a harmonic potential.

3.4. Higher potentials

For the class of trial functions we have considered, the general envelope equation in the presence of an external potential may be written

$$\frac{\phi_{xx}}{2\phi} - W(\phi) = \left[\frac{\ddot{\sigma}}{2\sigma}x^2 + \left(\ddot{x}_c - x_c\frac{\ddot{\sigma}}{\sigma}\right)x + V(x) + D(t)\right] \tag{28}$$

where $D(t)$ is a 'constant' of integration, as above. Solutions for harmonic and linear potentials is made possible by augmenting the slope of the linear potential or 'completing the square' in the quadratic case. For more general potentials, however, it is no longer possible to arrange the right-hand side into a quadratic form. Since we have already seen that, in the case of the CNLSE, consistent solution is not possible even for harmonic potentials, we must anticipate that in treating other nonlinear equations with potentials higher than quadratic, some difficulties may be encountered. For example, in the case of the LNLSE, we can show that consistent pulse solutions can be found for any potential supporting bound states. The difficulty we encounter is that the solutions we find will typically describe only *pinned states* since the propagation of a pulse through an anharmonic potential will generally require distortions of the pulse more general than the simple breathing motion we have considered.

Whenever the complex deformations of the pulse are not too strong, however, it is useful to approximate the true solution by 'breathing' envelopes such as (5). It may be expecting too much, however, to ask that the internal dynamics of such pulses be described by our 'breathing' equations ((20), (21), (24) and (26)) since the breathing equations should be affected more strongly by the neglected shape deformations than the basic envelope equation. However, some information can still be gleaned from our breathing equations in such cases where the external motion is sufficiently slow that the internal motion can essentially follow adiabatically. In this situation, the crucial consideration is that the acceleration $\ddot{\sigma}(t)$ should be at all times small relative to the other terms in the breathing equation. The value of the width is then determined straighforwardly by the instantaneous position of the centre of mass $x_c(t)$. Consequently, as the pulse propagates through a complex potential, its width does not oscillate independently, but varies in such a way as is required to conserve the total energy (approximately).

4. Conclusion

We have examined the conditions under which a family of pulse-shaped envelopes provide solutions to nonlinear Schrödinger equations in the presence of simple external potentials. As trial functions, we have chosen normalized pulses centred at $x = x_c(t)$ with widths proportional to $\sigma(t)$. Insertion of such trial shapes into a nonlinear Schrödinger equation leads to consistency conditions which are the equations of motion for the dynamical variables $x_c(t)$ and $\sigma(t)$. In the cases of constant and linearly varying potentials, we find that the dynamics of the width and the centre of mass are decoupled, allowing simultaneous independent translational motion and internal oscillations. For potentials having greater structure, however, the different nonlinear evolution equations produce different results. For the LNLSE, we continue to find simultaneous independent translational motion and internal oscillations in the presence of harmonic potentials, though the character of the internal oscillations is modified somewhat by the existence of the external potential. For the CNLSE, however, we find that the translational and internal motions become coupled to the point that consistent solution within the family of pulse shapes is not possible for any potential more complex than the linear one.

The class of functions we have treated here is certainly not the most general. While a variety of pulse shapes and frequencies of internal oscillation appear to be possible (e.g. for the LNLSE), all oscillations are of the breathing type, which essentially preserves the shape of our trial function at every point of the cycle. Shape oscillations requiring more complex deformations (rocking motions, for example) have been implicitly excluded.

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